THE DEGREE-DISTANCE OF SOME CLASS OF GRAPHS

S.Sujitha[1] and J.Jenisha[2]
Department of Mathematics
Holy Cross College(Autonomous)
Nagercoil-629004.

Abstract

Let $G = (V,E)$ be a finite connected graph. The degree-distance $D'(G)$ of a graph G is defined as $D'(G) = \sum_{\{u,v\}\subseteq V(G)}[deg(u) + deg(v)]d(u,v)$ where $deg(w)$ is the degree of the vertex w in G and $d(u, v)$ is the distance between u and v. In this paper, we determine the degree-distance $D'(G)$ for shadow graphs of complete graph, complete bipartite graph and study the relation between them. Also, it is shown that, the degree-distance of K_n is less than the degree-distance of shadow of K_n.

Keywords: distance, degree, degree-distance, shadow graph.

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1. Introduction

By a graph $G = (V, E)$, we mean a finite undirected connected graph. The *order* and *size* of G are denoted by n and p respectively. For vertices u and v in a connected graph G, the distance $d(u, v)$ is the length of a shortest $u - v$ path in G. The *degree* of a vertex v in a graph G is the number of edges of G incident with v and is denoted by $deg_G\, v$ or $deg\, v$. A graph G is *complete* if every two distinct vertices of G are adjacent. A complete graph of order n is denoted by K_n. A *bipartite graph* G is a graph whose vertex set $V(G)$ can be partitioned into two subsets V_1 and V_2 such that every edge of G joins V_1 with V_2; (V_1, V_2) is called a *bipartition* of G. If G contains every edge joining V_1 and V_2, then G is called a *complete bipartite graph*. The complete bipartite graph with bipartition (V_1, V_2) such that $|V_1| = m$ and $|V_2| = n$ is denoted by $K_{m,n}$. The shadow graph $S(G)$ of a connected graph G is constructed by taking two copies of G say G' and G''. Join each vertex u' in G' to the neighbours of the corresponding vertex u'' in G''. Let $G = (V,E)$ be a finite connected graph. The degree-distance $D'(G)$ of a graph G is defined as $D'(G) = \sum_{\{u,v\}\subseteq V(G)}[deg(u) + deg(v)]d(u,v)$ where $deg(w)$ is the degree of the vertex w in G and $d(u, v)$ is the distance between u and v. The union of two graphs $G_1 = (V_1, E_1)$ and

$G_2 = (V_2, E_2)$ is a graph $G(V,E)$ where $V = V_1 \cup V_2$ and $E = E_1 \cup E_2$ .The sum $G_1 + G_2$ is the graph $G_1 \cup G_2$ together with all the lines joining points of V_1 to the points of V_2 . The product $G_1 \times G_2$ as having $V = V_1 \times V_2$ and $u = (u_1, u_2)$ and $v = (v_1, v_2)$ are adjacent if $u_1 = v_1$ and u_2 is adjacent to v_2 in G_2 or u_1 is adjacent to v_1 in G_1 and $u_2 = v_2$. The line graph $L(G)$ of G is a graph in which the vertices are the lines of G and two points in $L(G)$ are adjacent iff the corresponding lines are adjacent in G. The degree distance seems to have been considered first by Dobrynin and Kochetova [1] and practically at the same time by Gutman [2], as a graph-theoretical descriptor for characterizing alkanes and then the same was studied by various authors. In the mathematical literature $D'(G)$ was investigated by Tomescu [4]. In this paper, we determine the degree-distance $D'(G)$ for shadow graphs of complete graph, complete bipartite graph and study the relation between them.

Throughout this paper G denotes a connected graph with at least two vertices. Our other notations are standard and taken mainly from [3]

Observation 1.1.[3] $K_m + K_n = K_{m+n}$

Observation 1.2. [3] $\overline{K_m} + \overline{K_n} = K_{m,n}$

2. Degree-Distance of Some Graphs

Theorem 2.1. For a complete graph of order n ,the degree distance is $n(n-1)^2$ where $n \geq 2$.

Proof.

Consider a complete graph K_n with n vertices ,

The degree distance of K_n is

$$D'(K_n) = \sum_{u,v \in V} [\deg u + \deg v]\, d(u,v)$$

$$= \sum_{u,v \in V} [n-1+n-1] \ (1)$$

$$= nC_2 \,.2(n-1)$$

Hence $D'(K_n) = n(n-1)^2$

Corollary 2.2. The degree distance of a sum of two complete graphs K_m and K_n is $(m+n)(m+n-1)^2$ for all $m, n \geq 1$

Proof.

The proof is obvious by observation 1.1.

Theorem 2.3. For a complete bipartite graph $K_{m,n}$,the degree distance is $mn(3m+3n-4)$

Proof.

$$D'\left(K_{m,n}\right) = \sum_{u,v\in V}[\deg u + \deg v]\, d(u,v)$$

$$= \sum_{u\in V_1, v\in V_2}[\deg u + \deg v]\, d(u,v) + \sum_{u,v\in V_1}[\deg u + \deg v]\, d(u,v) + + \sum_{u,v\in V_2}[\deg u + \deg v]\, d(u,v)$$

$$= \sum_{u\in V_1, v\in V_2}(n+m)\,(1) + \sum_{u,v\in V_1}(n+n)\,(2) + \sum_{u,v\in V_2}(m+m)\,(2)$$

$$= mn(m+n) + mC_2\,.\,4n + nC_2\,.\,4m$$

$$= mn(m+n) + 2mn(m-1) + 2mn(n-1)$$

Hence $D'\left(K_{m,n}\right) = mn(3m+3n-4)$

Corollary 2.4. $D'(\overline{K_m} + \overline{K_n}) = mn(3m+3n-4)$

Proof.

By observation 1.2, $D'(\overline{K_m} + \overline{K_n}) = mn(3m+3n-4)$

Therem 2.5. The degree distance of a product of two complete graphs K_m and K_n is $mn(m+n-2)(2mn-m-n))$.

Proof.

$$D'(K_m \times K_n) = \sum_{u,v\in V}[\deg u + \deg v]\, d(u,v)$$

$$= \sum_{u\in V_1, v\in V_2, x\in V_2}[\deg(u,v) + \deg(u,x)]\, d[(u,v),(u,x)] +$$

$$\sum_{u,v\in V_1, x\in V_2}[\deg(u,x) + \deg(v,x)]\, d[(u,x),(v,x)] +$$

$$\sum_{u,v\in V_1, x,y\in V_2}[\deg(u,x) + \deg(v,y)]\, d[(u,x),(v,y)]$$

$$= \sum_{u\in V_1, v\in V_2, x\in V_2}(m+n-2+m+n-2)(1) +$$

$$\sum_{u,v\in V_1, x\in V_2}(m+n-2+m+n-2)(1) +$$

$$\sum_{u,v\in V_1, x,y\in V_2}(m+n-2+m+n-2)(2)$$

$$= m\,.\,nC_2\,.2(m+n-2) + n\,.\,mC_2.\ 2(m+n-2) +$$

$$\frac{[n(m-1)(n-1)+n(m-2)(n-1)+\cdots+n(n-1)]\,.}{4(m+n-2)}$$

$$= mn(m+n-2)(n-1+m-1+2mn-2m-2n+2)$$

Hence $D'(K_m \times K_n) = mn(m+n-2)(2mn-m-n)$

Theorem 2.6. The degree distance of a line graph of a complete graph of n vertices is $(n-2)(n^4-4n^3+5n^2-2n)$.

Proof.

$$D'[L(K_n)] = \sum_{u,v\in V}[\deg u + \deg v]\, d(u,v)$$

$$= \sum_{u,v\in E(K_n)\&\, u,v\ adjacent\ in\ K_n}[\deg u + \deg v]\, d(u,v) + \sum_{u,v\in E(K_n)\&\, u,v\ not\ adjacent\ in\ K_n}[\deg u + \deg v]\, d(u,v)$$

$$= \sum_{u,v\in E(K_n)\&\, u,v\ adjacent}[2(n-2)+2(n-2)]1 + \sum_{u,v\in E(K_n)\&\, u,v\ not\ adjacent}[2(n-2)+2(n-2)]2$$

$$= \frac{nC_2}{2}\, 2(n-2)\,.4(n-2) + \frac{nC_2}{2}[nC_2 - 1 - 2(n-2)]\,.8(n-2)$$

$$= nC_2\,.4(n-2)^2 + 4\,.(nC_2)^2(n-2) - 4\,.nC_2(n-2) - 8\,.nC_2\,(n-2)^2$$

$$= (n-2)[2n^3-2n^2-4n^2+4n+n^4+n^2-2n^3-2n^2+2n-4n^3+12n^2-8n]$$

Hence $D'[L(K_n)] = (n-2)(n^4-4n^3+5n^2-2n)$

3 . Degree-Distance of Some Shadow Graphs

Theorem 3.1. The degree distance of a shadow of a complete graph of order n is $n(n-1)(7n-1)$ for $n \geq 3$

Proof.

$$D'[S(K_n)] = \sum_{u,v\in V}[\deg u + \deg v]\, d(u,v)$$

$$= \sum_{u,v\in V'}[\deg u + \deg v]\, d(u,v) + \sum_{u,v\in V''}[\deg u + \deg v]\, d(u,v) + \sum_{u\in V',v\in V''\,\&\,u,v\text{ are adjacent}}[\deg u + \deg v]\, d(u,v) + \sum_{u\in V',v\in V''\,\&\,u,v\text{ are not adjacent}}[\deg u + \deg v]\, d(u,v)$$

$$= \sum_{u,v\in V'}(2n-2+2n-2)1 + \sum_{u,v\in V''}(n-1+n-1)\,(2) + \sum_{u\in V',v\in V''\,\&\,u,v\text{ are adjacent}}(2n-2+n-1)1 + \sum_{u\in V',v\in V''\,\&\,u,v\text{ are not adjacent}}(2n-2+n-1)\,(2)$$

$$= nC_2\,.4(n-1) + nC_2\,.4(n-1) + n(n-1)\,.3(n-1) + n\,.6(n-1)$$

$$= (n-1)(4n^2-4n+3n^2+3n)$$

$$= (n-1)(7n^2-n)$$

Hence $D'[S(K_n)] = n(n-1)(7n-1)$

Remark 3.2. $D'[S(K_n)] = 28$ for $n = 2$

Remark 3.3. From theorem 2.1 and 3.1 , we can easily observe that ,the degree distance of a complete graph of order n is less than the degree distance of a shadow of complete graph of n vertices .

Theorem 3.4. The degree distance of a shadow of complete bipartite graph $K_{m,n}$ is

$20mn(m+n) - 12mn$.

Proof

$$D'[S(K_{m,n})] = \sum_{u,v\in V}[\deg u + \deg v]\, d(u,v)$$

$$= \sum_{u,v\in V_1'}[\deg u + \deg v]\, d(u,v) + \sum_{u,v\in V_2'}[\deg u + \deg v]\, d(u,v)$$

$$+ \sum_{u\in V_1',v\in V_2'}[\deg u + \deg v]\, d(u,v) + \sum_{u,v\in V_1''}[\deg u + \deg v]\, d(u,v) + \sum_{u,v\in V_2''}[\deg u + \deg v]\, d(u,v) + \sum_{u\in V_1'',v\in V_2''}[\deg u + \deg v]\, d(u,v) + \sum_{u\in V_1',v\in V_1''}[\deg u + \deg v]\, d(u,v) + \sum_{u\in V_2',v\in V_2''}[\deg u + \deg v]\, d(u,v) + \sum_{u\in V_1',v\in V_2''}[\deg u + \deg v]\, d(u,v) + \sum_{u\in V_2',v\in V_1''}[\deg u + \deg v]\, d(u,v)$$

$$= \sum_{u,v \in V_1{}'} (2n+2n)\,(2) + \sum_{u,v \in V_2{}'} (2m+2m)\,(2)$$

$$+ \sum_{u \in V_1{}', v \in V_2{}'} (2n+2m)(1) + \sum_{u,v \in V_1{}''} \ (n+n)\,(2)$$

$$+ \sum_{u,v \in V_2{}''} (m+m)(2) + \sum_{u \in V_1{}'', v \in V_2{}''} (n+m)\,(3)$$

$$+ \sum_{u \in V_1{}', v \in V_1{}''} (2n+n)(2) + \sum_{u \in V_2{}', v \in V_2{}''} (2m+m)\,(2)$$

$$+ \ \sum_{u \in V_1{}', v \in V_2{}''} (2n+m)(1) + \sum_{u \in V_2{}', v \in V_1{}''} (2m+n)(1)$$

$$= mC_2\,.8n + nC_2\,.8m + 2mn(n+m) + mC_2\,.4n + nC_2\,.4m + 3mn(n+m) + 6m^2n + 6n^2m + mn(2n+m) + mn(2m+n)$$

$$= 6[mn(m-1) + mn(n-1)] + 14mn(m+n)$$

$$= 6mn\big(m-1+n-1) + 14mn(m+n)\big)$$

$$= 4mn(5m+5n-3)$$

Hence $D'\big[S\big(K_{m,n}\big)\big] \ = 20mn(m+n) - 12mn$

Remark 3.5. From theorem 2.3 and 3.4 , we can easily observe that ,the degree distance of a complete bipartite graph is less than the degree distance of a shadow of a complete bibartite graph .

Theorem 3.6. $D'[S(K_m \times K_n)] = 4(m+n-2)[2m\,.nC_2 + 2n\,.mC_2 \ + mn(n-1)(m-1)]$

$$+3(m+n-2)[2mn + n(n-1) + m(m-1) + 2mn(n-1)(m-1)]$$

Proof.

$$D'[S(K_m \times K_n)] = \sum_{u,v \in V} [\deg u + \deg v] \ d(u,v)$$

$$= \sum_{u \in V_1{}', v \in V_2{}', x \in V_2{}'} [\deg(u,v) + \deg(u,x)] \ d[(u,v),(u,x)] +$$

$$\sum_{u,v \in V_1{}', x \in V_2{}'} [\deg(u,x) + \deg(v,x)] \ d[(u,x),(v,x)] +$$

$$\sum_{u,v \in V_1{}', x,y \in V_2{}'} [\deg(u,x) + \deg(v,y)] \ d[(u,x),(v,y)] +$$

$$\sum_{u \in V_1{}'', v,x \in V_2{}'',} [\deg(u,v) + \deg(u,x)] \ d[(u,v),(u,x)] +$$

$$\sum_{u,v\in V_1'',x\in V_2''}[\deg(u,x)+\deg(v,x)]\,d[(u,x),(v,x)]+$$

$$\sum_{u,v\in V_1'',x,y\in V_2''}[\deg(u,x)+\deg(v,y)]\,d[(u,x),(v,y)]+$$

$$\sum_{u\in V_1',u'\in V_1'',v\in V_2',v'\in V_2''}[\deg(u,v)+\deg(u',v')]\,d[(u,v),(u',v')]+$$

$$\sum_{u\in V_1',u'\in V_1'',v\in V_2',x\in V_2''}[\deg(u,v)+\deg(u',x)]\,d[(u,v),(u',x)]+$$

$$\sum_{u\in V_1',x\in V_1'',v\in V_2',v'\in V_2''}[\deg(u,v)+\deg(x,v')]\,d[(u,v),(x,v')]+$$

$$\sum_{u\in V_1,'x'\in V_1'',v\in V_2',y'\in V_2''}[\deg(u,v)+\deg(x',y')]\,d[(u,v),(x',y')]$$

$$=\sum_{u\in V_1',v\in V_2',x\in V_2'}(2m+2n-4+2m+2n-4)1+$$

$$\sum_{u,v\in V_1',x\in V_2'}(2m+2n-4+2m+2n-4)1+$$

$$\sum_{u,v\in V_1',x,y\in V_2'}(2m+2n-4+2m+2n-4)2+$$

$$\sum_{u\in V_1'',v,x\in V_2''}(m+n-2+m+n-2)3+$$

$$\sum_{u,v\in V_1'',x\in V_2''}(m+n-2+m+n-2)2+$$

$$\sum_{u,v\in V_1'',x,y\in V_2''}(m+n-2+m+n-2)2+$$

$$\sum_{u\in V_1',u'\in V_1'',v\in V_2',v'\in V_2''}(2m+2n-4+m+n-2)2+$$

$$\sum_{u\in V_1',u'\in V_1'',v\in V_2',x'\in V_2''}(2m+2n-4+m+n-2)1+$$

$$\sum_{u\in V_1',x'\in V_1'',v\in V_2',v'\in V_2''}(2m+2n-4+m+n-2)1\ +$$

$$\sum_{u\in V_1,'x'\in V_1'',v\in V_2',y'\in V_2''}(2m+2n-4+m+n-2)2$$

$$=m\,.\,nC_2\,.4(m+n-2)+n\,.\,mC_2\,.\,4(m+n-2)$$

$$+n(n-1)(m-1+m-2+\cdots+1)\,.\,8(m+n-2)$$

$$+m\,.\,nC_2\,.4(m+n-2)+n\,.\,mC_2\,.\,4(m+n-2)$$

$$+n(n-1)(m-1+m-2+\cdots+1)\,.\,6(m+n-2)$$

$$+mn\,.6(m+n-2)+n(n-1)\,.3(m+n-2)$$

$$+m(m-1).3(m+n-2)+n(n-1)(m-1+m-2+\cdots+1)\,.6(m+n-2)$$

Hence $D'[S(K_m \times K_n)] = 4(m+n-2)[2m\,.nC_2 + 2n\,.mC_2 \ + mn$

$$(n-1)(m-1)] + 3(m+n-2)[2mn + n(n-1)$$

$$+m(m-1) + \ 2mn(n-1)(m-1)]$$

Theorem 3.7. The degree distance of a shadow of the line graph of a complete graph with n vertices is $n\,(n-2)\,[6n^3 - 22n^2 + 30n - 14]$

Proof.

$$D'\left[S\left(L(K_n)\right)\right] = \sum_{u,v\in V}[\deg u + \deg v]\, d(u,v)$$

$$= \sum_{u,v\in E'(K_n)\&\, u,v\ adjacent} [\deg u + \deg v]\, d(u,v) +$$

$$\sum_{u,v\in E'(K_n)\&\, u,v\ not\ adjacent} [\deg u + \deg v]\, d(u,v) +$$

$$\sum_{u\in E'(K_n), v\in E''(K_n)\&\, u,v\ adjacent} [\deg u + \deg v]\, d(u,v) +$$

$$\sum_{u\in E'(K_n), v\in E''(K_n)\&\, u,v\ not\ adjacent} [\deg u + \deg v]\, d(u,v) +$$

$$\sum_{u,v\in E''(K_n)} [\deg u + \deg v]\, d(u,v)$$

$$= \sum_{u,v\in E'(K_n)\&\, u,v\ adjacent} [4(n-2) + 4(n-2)]1 +$$

$$\sum_{u,v\in E'(K_n)\&\, u,v\ not\ adjacent} [4(n-2) + 4(n-2)]2 +$$

$$\sum_{u\in E'(K_n), v\in E''(K_n)\&\, u,v\ adjacent} [4(n-2) + 2(n-2)]1 +$$

$$\sum_{u\in E'(K_n), v\in E''(K_n)\&\, u,v\ not\ adjacent} [4(n-2) + 2(n-2)]2 +$$

$$\sum_{u,v\in E''(K_n)} [2(n-2) + 2(n-2)]2$$

$$= \frac{nC_2}{2}\,.2(n-2).8(n-2) + \frac{nC_2}{2}[nC_2 - 1 - 2(n-2)]\,.16(n-2) +$$

$$nC_2 .2(n-2).6(n-2) + \{nC_2[nC_2 - 1 - 2(n-2)] + nC_2\} .12(n-2) + (nC_2)C_2 .8(n-2)$$

$$= 10(n-2)\left[\frac{n^4}{2} - 2n^3 + \frac{5n^2}{2} - n\right] + 6n(n-1)(n-2) + 8.(nC_2)C_2 .(n-2)$$

$$= (n-2)\,[5n^4 - 20n^3 + 25n^2 - 10n + 6n^2 - 6n + n^2(n-1)^2 - 2n(n-1)]$$

$$= (n-2)[5n^4 - 20n^3 + 31n^2 - 16n + n^4 - 2n^3 + n^2 - 2n^2 + 2n]$$

Hence $D'\left[S\left(L(K_n)\right)\right] = n\,(n-2)\,[6n^3 - 22n^2 + 30n - 14]$

4.Conclusion.

In this paper we obtained the degree distance of a complete graph ,the sum of two complete graphs,the sum of a complement of two complete graphs , the line graph of a complete graph and some shadow graphs. Also it is shown that the degree distance of the complete graph is less than the degree distance of a shadow of a complete graph.

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